CEE 271 APPLIED MECHANICS II

Lectures 4 & 5: Curvilinear Motion: Normal and Tangential Components & Cylindrical Components

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Today's Objectives

- Determine the normal and tangential components of velocity and acceleration of a particle traveling along a curved path.
- Determine velocity and acceleration components using cylindrical coordinates.

Outline (Pre-Job Brief)

- Normal and Tangential Components
  - Velocity and Acceleration
- Cylindrical Components
  - Velocity and Acceleration
- Examples and Questions
- Summary and Feedback

Curvilinear Motion: Normal and Tangential Components

Applications

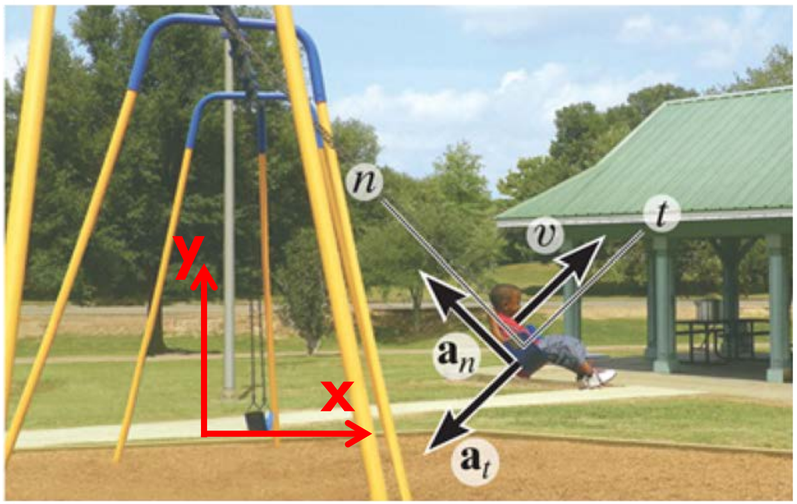
University of Hawai'i Mānoa

Cars traveling along a clover-leaf interchange experience an acceleration due to a change in velocity as well as due to a change in direction of the velocity.

If the car's speed is increasing at a known rate as it travels along a curve, how can we determine the magnitude and direction of its total acceleration?

Why would you care about the total acceleration of the car?

Applications (continued)



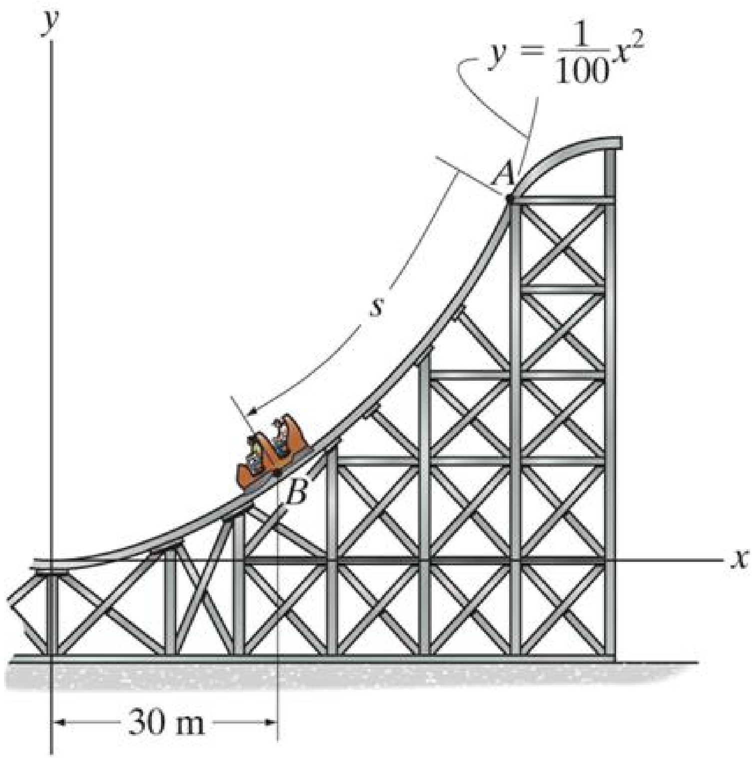
As the boy swings upward with a velocity **v**, his motion can be analyzed using *n–t* coordinates.

As he rises, the magnitude of his velocity is changing, and acceleration as well.

How can we determine his velocity and acceleration at the bottom of the arc?

Can we use different coordinates, such as x-y coordinates, to describe his motion? Which coordinate system would be easier to use to describe his motion? Why ?

Applications (continued)



A roller coaster travels down a hill for which the path can be approximated by a function y = f(x).

The roller coaster starts from rest and increases its speed at a constant rate.

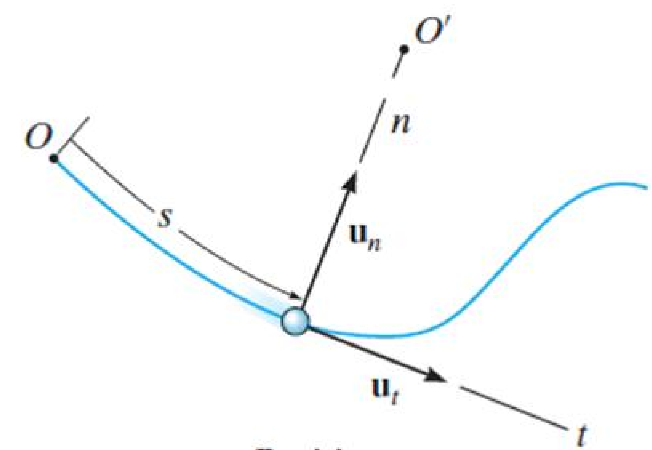
How can we determine its velocity and acceleration at the bottom?

Why would we want to know these values?

Normal and Tangential Components

When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian. When the path of motion is known, normal (n) and tangential (t) coordinates are often used.

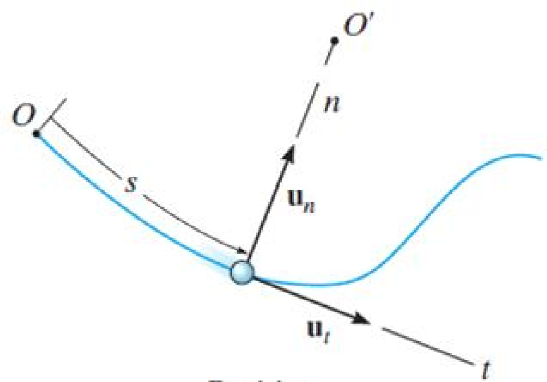
In the n-t coordinate system, the origin is located on the particle (the origin moves with the particle).



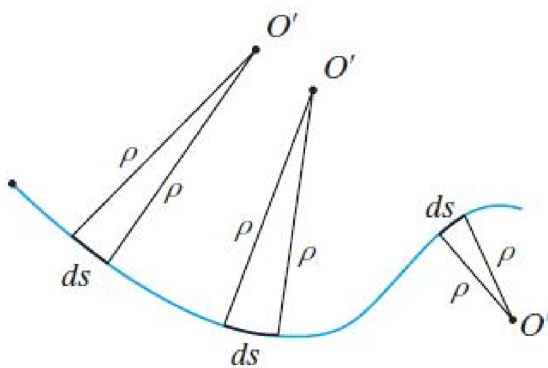
The t-axis is tangent to the path (curve) at the instant considered, positive in the direction of the particle's motion.

The n-axis is perpendicular to the t-axis with the positive direction toward the center of curvature of the curve.

Normal and Tangential Components



Position



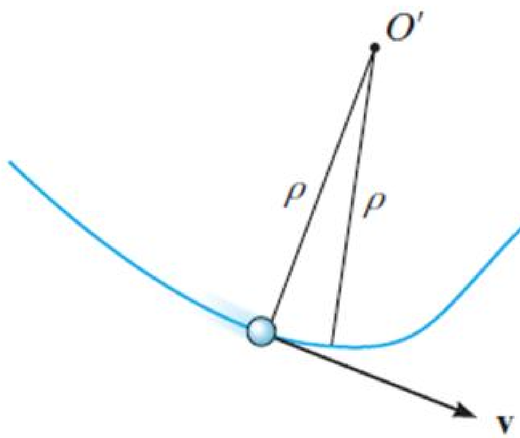
Radius of curvature

The positive n and t directions are defined by the unit vectors $\boldsymbol{u}_n$ and $\boldsymbol{u}_t$, respectively.

The center of curvature, O', always lies on the concave side of the curve. The radius of curvature, ρ, is defined as the perpendicular distance from the curve to the center of curvature at that point.

The position of the particle at any instant is defined by the distance, s, along the curve from a fixed reference point.

Velocity in the n-t Coordinate System



Velocity

The velocity vector is always tangent to the path of motion (t-direction).

The magnitude is determined by taking the time derivative of the path function, s(t).

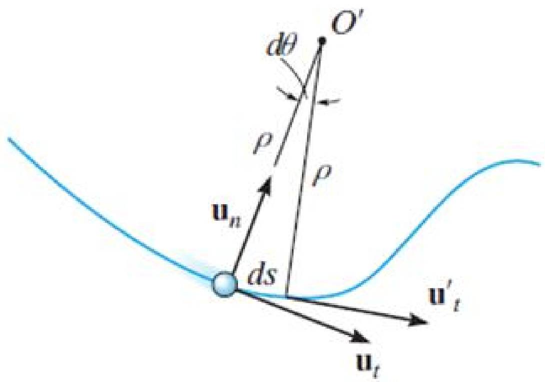
$$\boldsymbol{v} = v\, \boldsymbol{u}_t \quad \text{where} \quad v = \dot{s} = ds/dt$$

Here v defines the magnitude of the velocity (speed) and $\boldsymbol{u}_t$ defines the direction of the velocity vector.

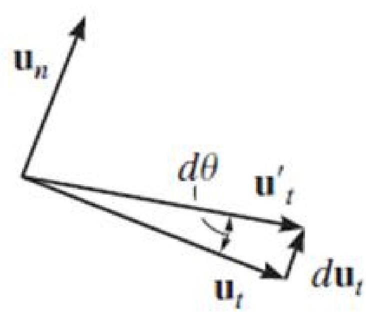
Acceleration in the n-t Coordinate System

Acceleration is the time rate of change of velocity:

$$\mathbf{a} = d\boldsymbol{v}/dt = d(v\boldsymbol{u}_t)/dt = \dot{v}\boldsymbol{u}_t + v\dot{\boldsymbol{u}}_t$$



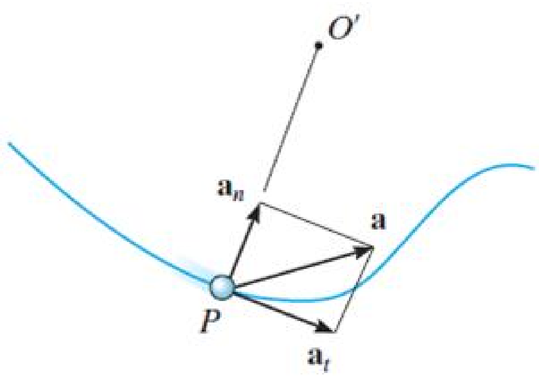
Here $\dot{v}$ represents the change in the magnitude of velocity and $\dot{\boldsymbol{u}}_t$ represents the rate of change in the direction of $\boldsymbol{u}_t$.



After mathematical manipulation, the acceleration vector can be expressed as:

$$\mathbf{a} = \dot{v}\, \boldsymbol{u}_t + (v^2/\rho)\, \boldsymbol{u}_n = a_t\, \boldsymbol{u}_t + a_n\, \boldsymbol{u}_n.$$

Acceleration in the n-t Coordinate System



Acceleration

So, there are two components to the acceleration vector:

$$\mathbf{a} = a_t \, \boldsymbol{u}_t + a_n \, \boldsymbol{u}_n$$

- The tangential component is tangent to the curve and in the direction of increasing or decreasing velocity.

$$a_t = \dot{v} \quad \text{or} \quad a_t \, ds = v \, dv$$

- The normal or centripetal component is always directed toward the center of curvature of the curve. $a_n = v^2/\rho$
- The magnitude of the acceleration vector is

$$a = [(a_t)^2 + (a_n)^2]^{0.5}$$

Special Cases of Motion

There are some special cases of motion to consider.

1) The particle moves along a straight line.

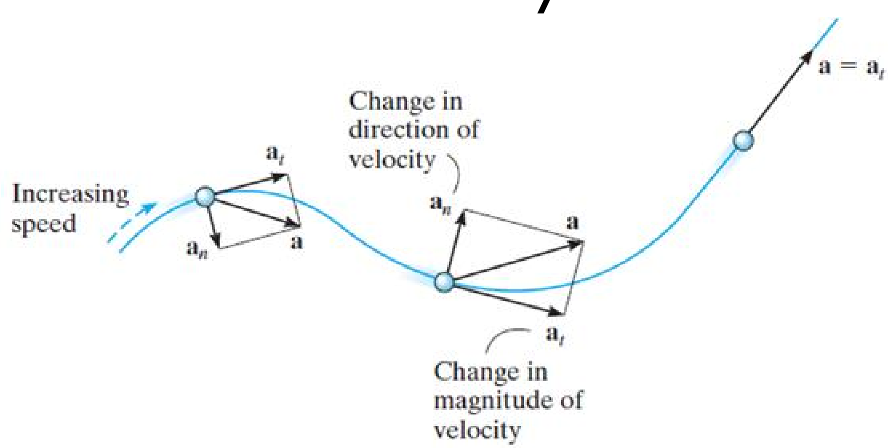
   $\rho \longrightarrow \infty \quad => \quad a_n = v^2/\rho = 0 \quad => \quad a = a_t = \dot{v}$

   The tangential component represents the time rate of change in the magnitude of the velocity.

2) The particle moves along a curve at constant speed.

   $a_t = \dot{v} = 0 \quad => \quad a = a_n = v^2/\rho$

   The normal component represents the time rate of change in the direction of the velocity.



Special Cases of Motion (continued)

3) The tangential component of acceleration is constant, $a_t = (a_t)_c$. In this case,

$$s = s_o + v_o t + (1/2)\,(a_t)_c\, t^2$$

$$v = v_o + (a_t)_c\, t$$

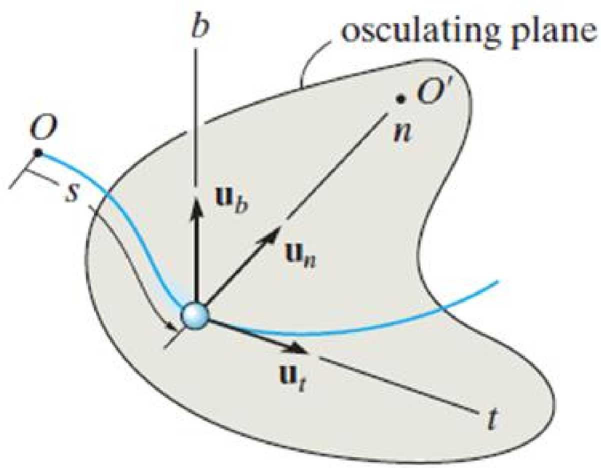
$$v^2 = (v_o)^2 + 2\,(a_t)_c\,(s - s_o)$$

As before, s_o and v_o are the initial position and velocity of the particle at $t = 0$.

4) The particle moves along a path expressed as $y = f(x)$. The radius of curvature, ρ, at any point on the path can be calculated

$$\rho = \frac{[\,1 + (dy/dx)^2\,]^{3/2}}{\left| d^2y/dx^2 \right|}$$

Three-Dimensional Motion



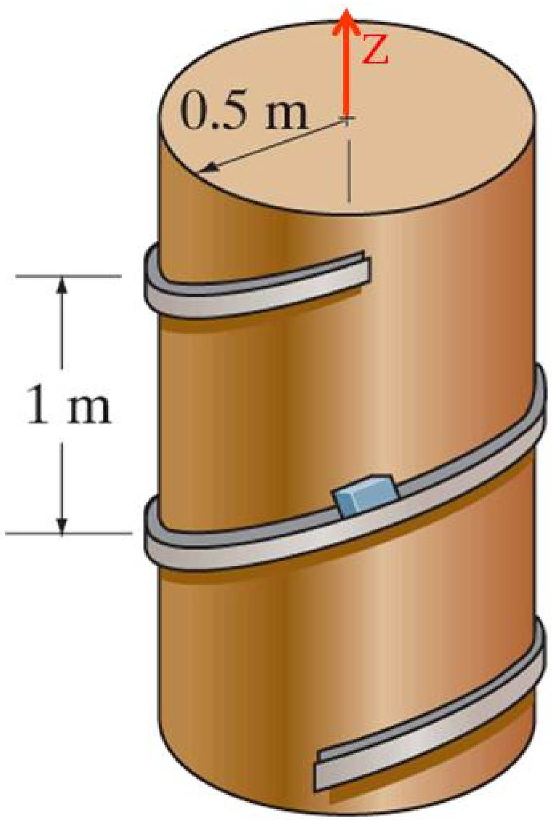
If a particle moves along a space curve, the n and t axes are defined as before. At any point, the t-axis is tangent to the path and the n-axis points toward the center of curvature. The plane containing the n and t axes is called the osculating plane.

A third axis can be defined, called the binomial axis, b. The binomial unit vector, $\boldsymbol{u}_b$, is directed perpendicular to the osculating plane, and its sense is defined by the cross product $\boldsymbol{u}_b = \boldsymbol{u}_t \times \boldsymbol{u}_n$.

There is no motion, thus no velocity or acceleration, in the binomial direction.

Curvilinear Motion: Cylindrical Components

Applications



A cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the box slides down the helical ramp. How would you find the box's velocity components to check to see if the package will fly off the ramp?

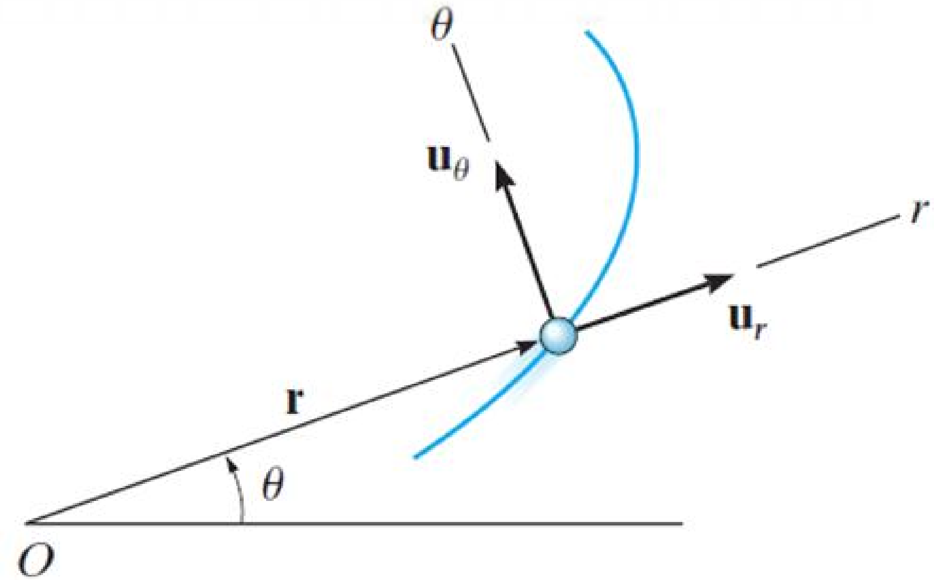
Applications (continued)

The cylindrical coordinate system can be used to describe the motion of the girl on the slide.

Here the radial coordinate is constant, the transverse coordinate increases with time as the girl rotates about the vertical axis, and her altitude, z, decreases with time.

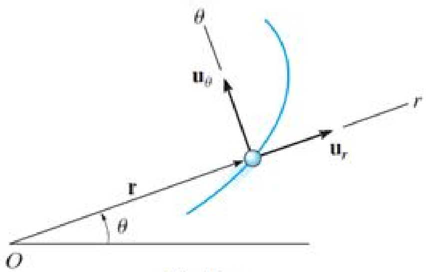
How can you find her acceleration components ?

Cylindrical Components

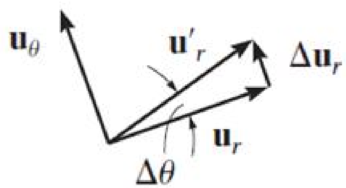


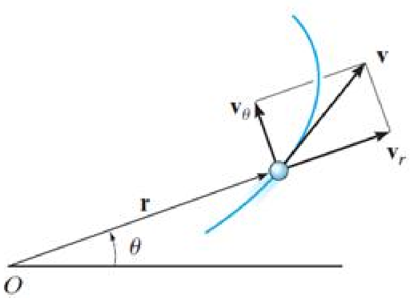
We can express the location of P in polar coordinates as $\boldsymbol{r} = r\,\boldsymbol{u}_r$. Note that the radial direction, r, extends outward from the fixed origin, O, and the transverse coordinate, θ, is measured counter-clockwise (CCW) from the horizontal.

Velocity in Polar Coordinates



Position





Velocity

The instantaneous velocity is defined as:

$$\mathbf{v} = d\mathbf{r}/dt = d(r\mathbf{u}_r)/dt$$

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt}$$

Using the chain rule:

$$d\mathbf{u}_r/dt = (d\mathbf{u}_r/d\theta)(d\theta/dt)$$

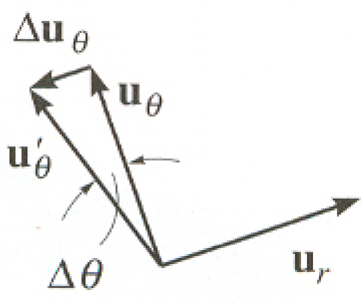
We can prove that $d\mathbf{u}_r/d\theta = \mathbf{u}_\theta$ so $d\mathbf{u}_r/dt = \dot{\theta}\mathbf{u}_\theta$

Therefore: $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$

Thus, the velocity vector has two components: $\dot{r}$, called the radial component, and $r\dot{\theta}$ called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or

$$v = \sqrt{(r\,\dot{\theta})^2 + (\,\dot{r}\,)^2}$$

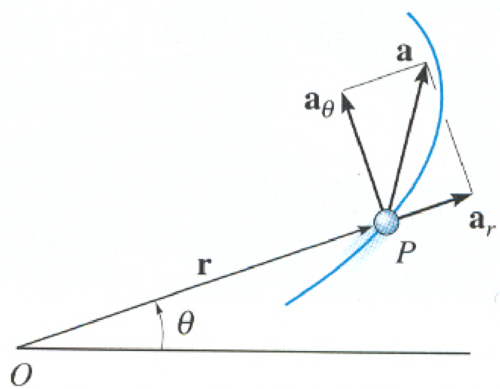
Acceleration (Polar Coordinates)



The instantaneous acceleration is defined as:

$$\mathbf{a} = d\mathbf{v}/dt = (d/dt)(\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta)$$

After manipulation, the acceleration can be expressed as



Acceleration

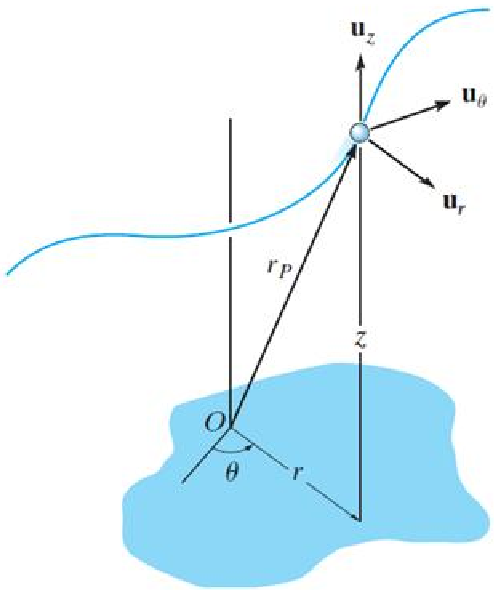
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$

The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or a_r .

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or a_θ.

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$

Cylindrical Coordinates



If the particle P moves along a space curve, its position can be written as

$$\boldsymbol{r}_P = r\boldsymbol{u}_r + z\boldsymbol{u}_z$$

Taking time derivatives and using the chain rule:

Velocity: $\boldsymbol{v}_P = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta + \dot{z}\boldsymbol{u}_z$

Acceleration: $\mathbf{a}_P = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta + \ddot{z}\boldsymbol{u}_z$

Examples & Questions

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