CEE 271 APPLIED MECHANICS II

Lecture 3: Curvilinear Motion and Motion of a Projectile

Department of Civil & Environmental Engineering
University of Hawaiʻi at Mānoa

Today's Objectives

- Describe the motion of a particle traveling along a curved path.
- Relate kinematic quantities in terms of the rectangular components of the vectors.
- Analyze the free-flight motion of a projectile.

Outline (Pre-Job Brief)

- Brief Review
- General Curvilinear Motion
- Rectangular Components
- Motion of a Projectile
- Examples and Questions
- Summary and Feedback

Review – Vector Analysis

- A scalar has magnitude
- A vector has direction and magnitude
- Vector addition can be performed geometrically or algebraically

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} B_x \\ B_y \end{bmatrix} + \begin{bmatrix} C_x \\ C_y \end{bmatrix} = \begin{bmatrix} B_x + C_x \\ B_y + C_y \end{bmatrix}$$

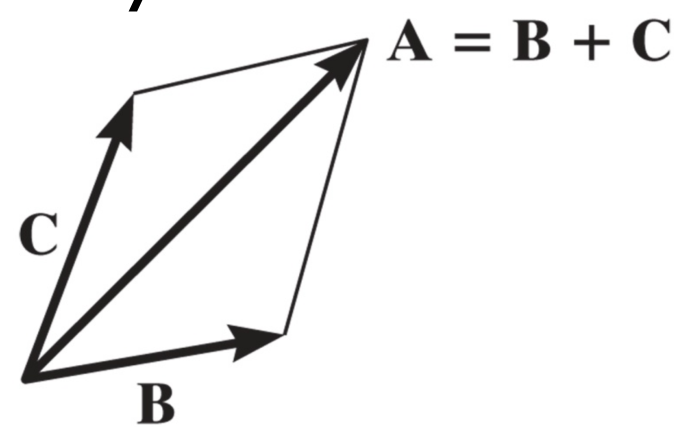


Figure: AppB_001

Review – Vector Analysis

- A Unit Vector has direction and unit magnitude
- Given a vector $\vec{\mathbf{A}}$, with magnitude A, the unit vector in the direction of $\vec{\mathbf{A}}$ is

$$u_A = \frac{\vec{A}}{A}$$

- In Cartesian coordinates, $\vec{x}$, $\vec{y}$ and $\vec{z}$, unit vectors in each positive axis direction are labeled **i**, **j**, and **k**, respectively.

Review – Vector Analysis

- A 3-D Vector can be represented by a combination of the Cartesian unit vectors:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

- The magnitude of A is:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- The unit vector in the direction of A is then:

$$u_A = \frac{A_x}{A}i + \frac{A_y}{A}j + \frac{A_z}{A}k$$

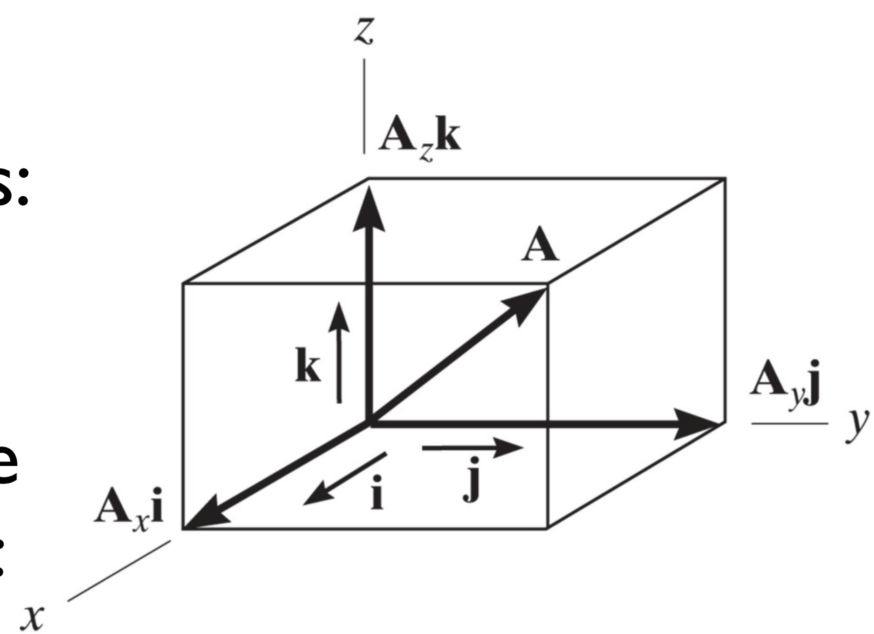


Figure: AppB_002

Review – Vector Analysis

- Given the angles between vector A and each axis:

$$\cos\alpha = \frac{A_x}{A},\ \cos\beta = \frac{A_y}{A},\ \cos\gamma = \frac{A_z}{A}$$

- Therefore the unit vector is:

$$u_A = \frac{A_x}{A} i + \frac{A_y}{A} j + \frac{A_z}{A} k$$

or

$$u_A = \cos\alpha\ i + \cos\beta\ j + \cos\gamma\ k$$

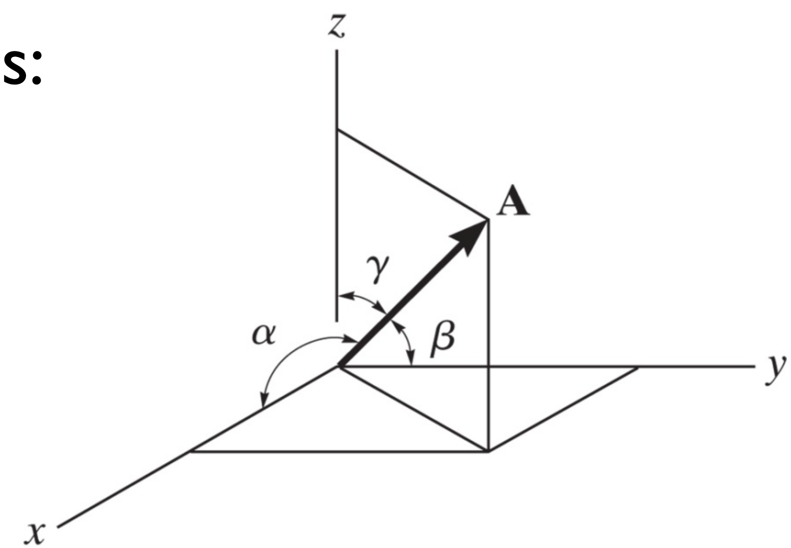
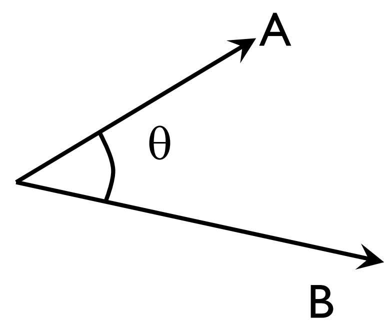


Figure: AppB_003

Review – Vector Analysis

- The dot product of two vectors, A and B, which form an angle θ, is a scalar defined as:

$$A \bullet B = AB\cos\theta$$



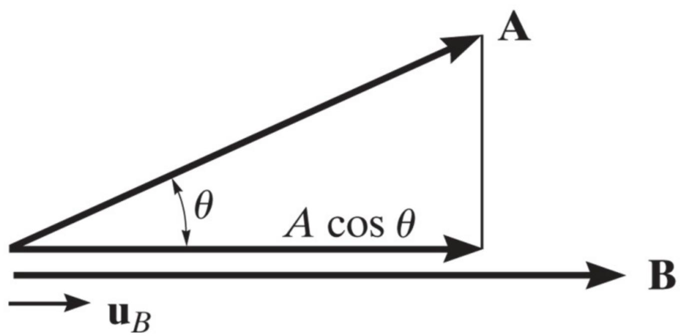
- In Cartesian Coordinates:

$$A \bullet B = A_x B_x + A_y B_y + A_z B_z$$

Review – Vector Analysis

- We can use the dot product to find the component of a vector in a given direction:

$$A \bullet u_B = A \times 1 \cos\theta = A\cos\theta$$



- The dot product is commutative:

$$A \bullet B = B \bullet A$$

- And distributive:

$$A \bullet (B + C) = A \bullet B + A \bullet C$$

Review – Vector Analysis

- The cross product of two vectors yields a resultant vector which is perpendicular to the plane of the two vectors:

$$C = A \times B$$

- The cross product observes the right-hand rule, therefore:

$$A \times B = -B \times A$$

- The magnitude of C is:

$$C = AB \sin \theta$$

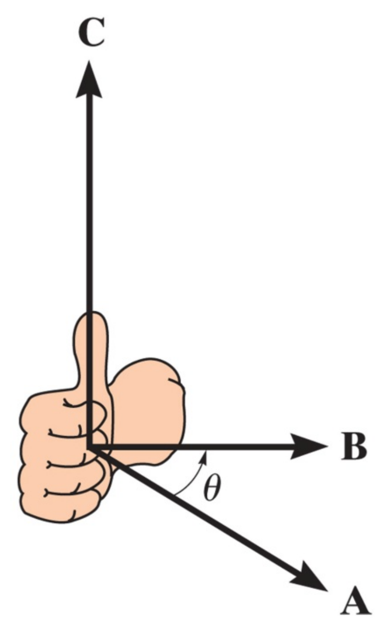


Figure: AppB_004

Review – Vector Analysis

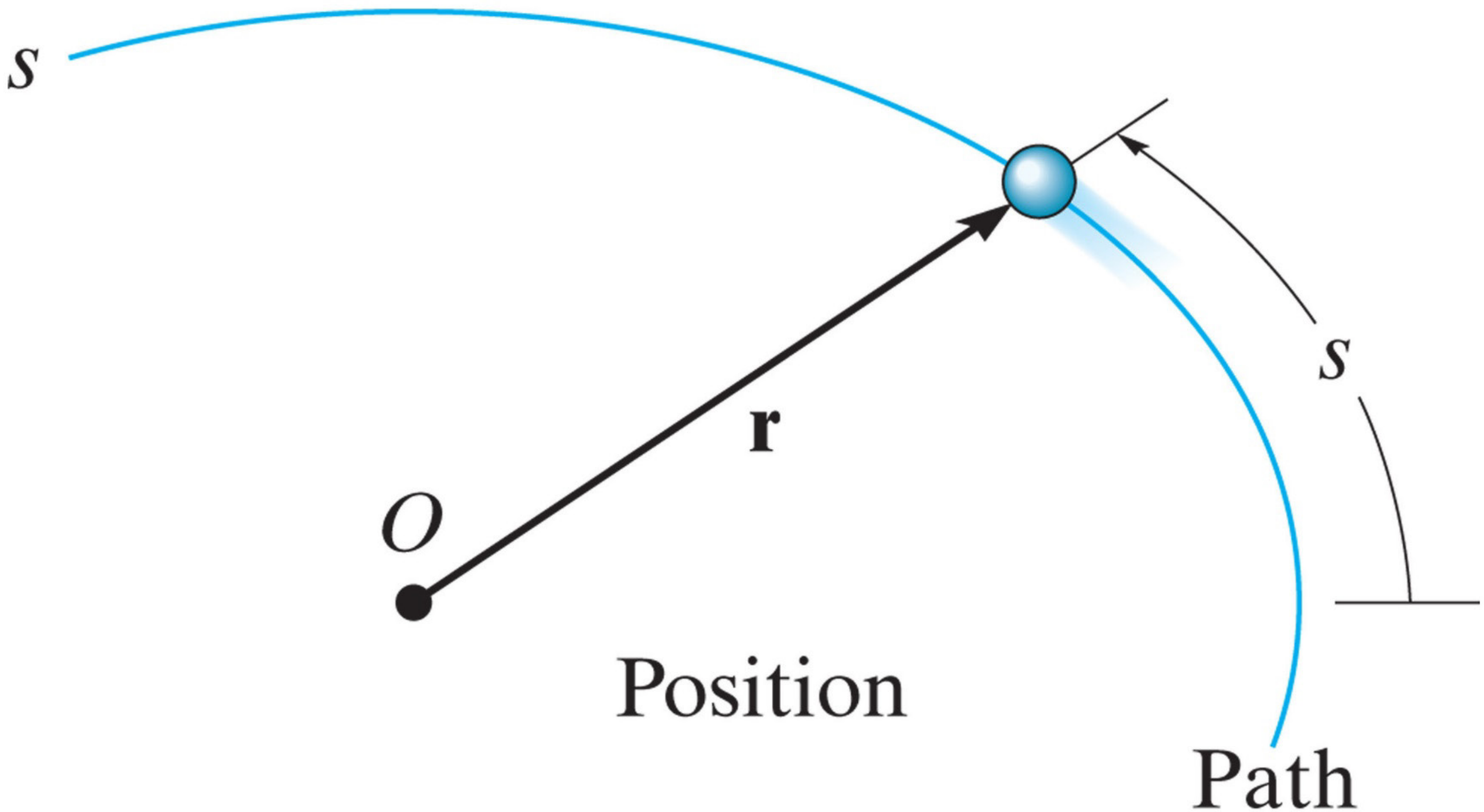
- If A and B are expressed in Cartesian component form, the cross product can be evaluated by expanding the determinant as follows:

$$C = A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Which gives:

$$C = (A_y B_z - A_z B_y) i - (A_x B_z - A_z B_x) j + (A_x B_y - A_y B_x) k$$

Curvilinear Motion

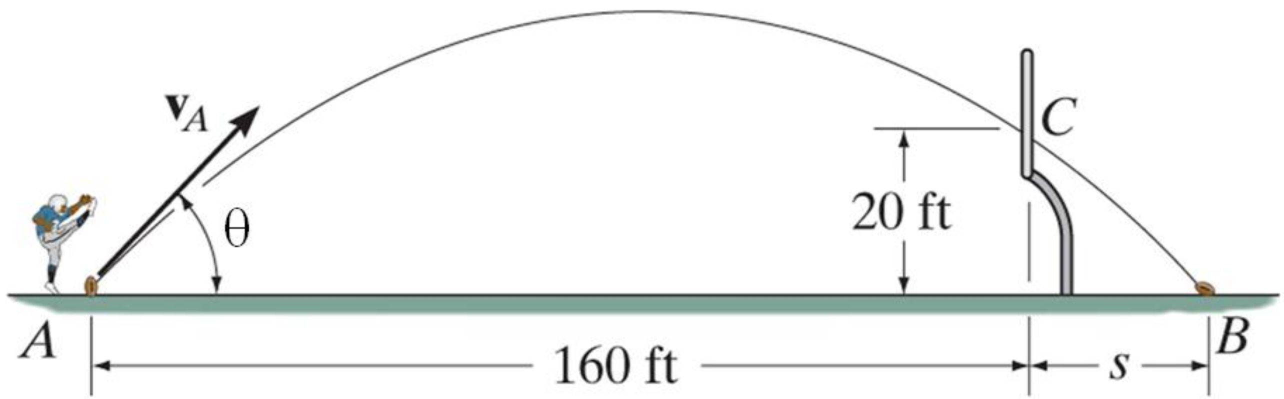


Applications

The path of motion of a plane can be tracked with radar and its x, y, and z coordinates (relative to a point on earth) recorded as a function of time.

How can we determine the velocity or acceleration of the plane at any instant?

Applications (continued)



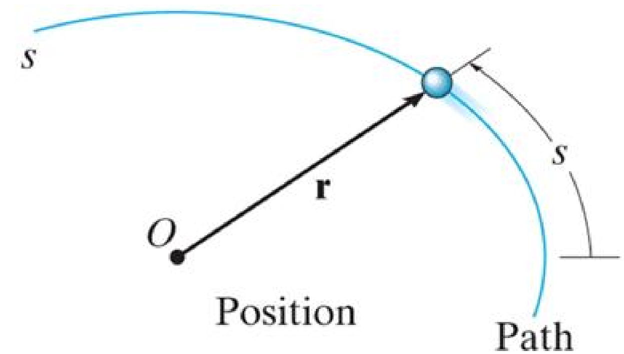
A good kicker instinctively knows at what angle, θ, and initial velocity, v_A, he must kick the ball to make a field goal.

For a given kick "strength", at what angle should the ball be kicked to get the maximum distance?

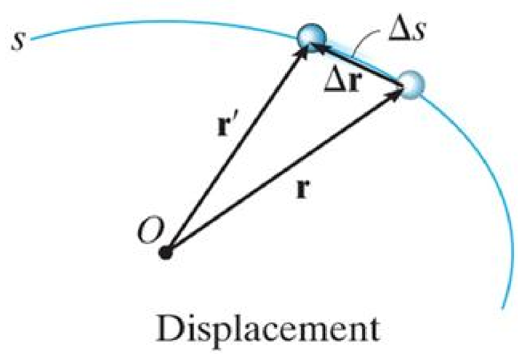
General Curvilinear Motion

A particle moving along a curved path undergoes curvilinear motion. Since the motion is often three-dimensional, vectors are used to describe the motion.

A particle moves along a curve defined by the path function, s.



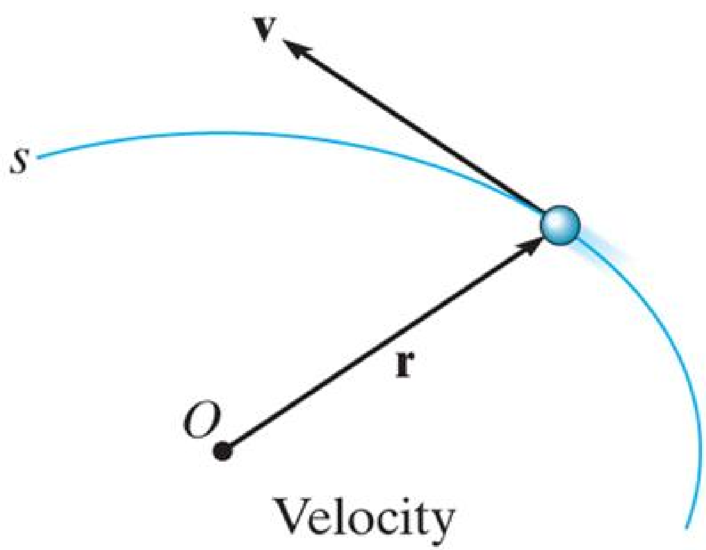
The position of the particle at any instant is designated by the vector ***r*** = ***r***(t). Both the magnitude and direction of ***r*** may vary with time.



If the particle moves a distance Δs along the curve during time interval Δt, the displacement is determined by vector subtraction: Δ***r*** = ***r’*** - ***r***

Velocity

Velocity represents the rate of change in the position of a particle.



The average velocity of the particle during the time increment Δt is $\boldsymbol{v}_{avg} = \Delta \boldsymbol{r}/\Delta t$.

The instantaneous velocity is the time-derivative of position $\boldsymbol{v} = d\boldsymbol{r}/dt$.

The velocity vector, ***v***, is always tangent to the path of motion.

The magnitude of ***v*** is called the speed. Since the arc length Δs approaches the magnitude of $\Delta \boldsymbol{r}$ as $t \rightarrow 0$, the speed can be obtained by differentiating the path function ($v = ds/dt$). Note that this is not a vector!

Acceleration

Acceleration represents the rate of change in the velocity of a particle.

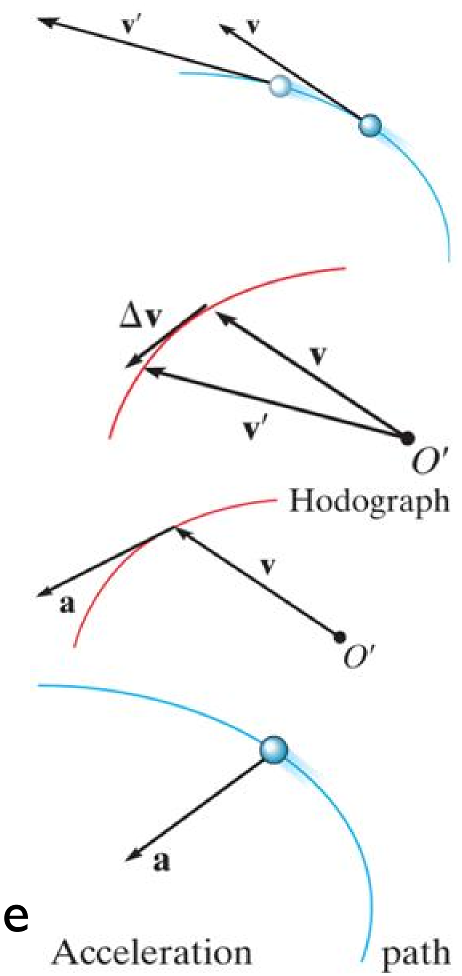
If a particle's velocity changes from **v** to **v'** over a time increment Δt, the average acceleration during that increment is:

$$\mathbf{a}_{avg} = \Delta\mathbf{v}/\Delta t = (\mathbf{v} - \mathbf{v}')/\Delta t$$

The instantaneous acceleration is the time-derivative of velocity:

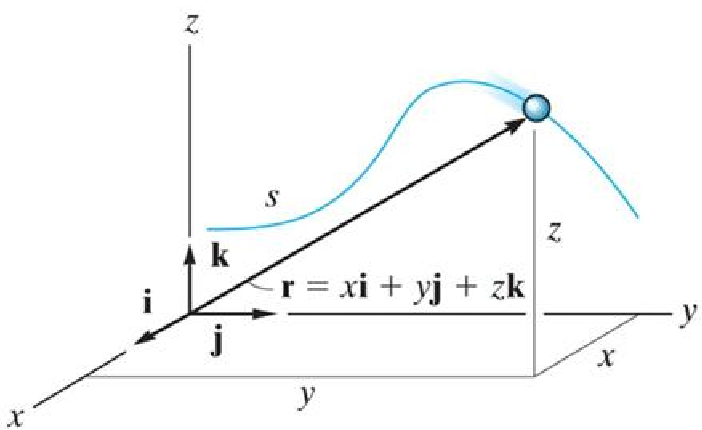
$$\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2$$

A plot of the locus of points defined by the arrowhead of the velocity vector is called a hodograph. The acceleration vector is tangent to the hodograph, but not, in general, tangent to the path function.



Rectangular Components

It is often convenient to describe the motion of a particle in terms of its x, y, z or rectangular components, relative to a fixed frame of reference.



Position

The position of the particle can be defined at any instant by the position vector

$$\boldsymbol{r} = \boldsymbol{x}\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\ .$$

The x, y, z components may all be functions of time, i.e., $x = x(t)$, $y = y(t)$, and $z = z(t)$.

The magnitude of the position vector is: $r = (x^2 + y^2 + z^2)^{0.5}$

The direction of $\boldsymbol{r}$ is defined by the unit vector: $\boldsymbol{u_r} = (1/r)\boldsymbol{r}$

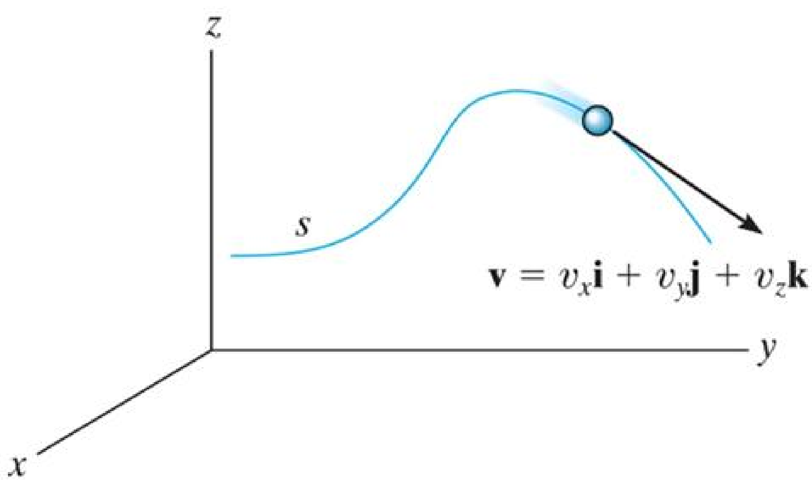
Rectangular Components: Velocity

The velocity vector is the time derivative of the position vector:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(x\boldsymbol{i})/dt + d(y\boldsymbol{j})/dt + d(z\boldsymbol{k})/dt$$

Since the unit vectors ***i***, ***j***, ***k*** are constant in magnitude and direction, this equation reduces to $v = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$

where $v_x = \dot{x} = dx/dt$, $v_y = \dot{y} = dy/dt$, $v_z = \dot{z} = dz/dt$



Velocity

The magnitude of the velocity vector is

$$v = [(v_x)^2 + (v_y)^2 + (v_z)^2]^{0.5}$$

The direction of ***v*** is tangent to the path of motion.

Rectangular Components: Acceleration

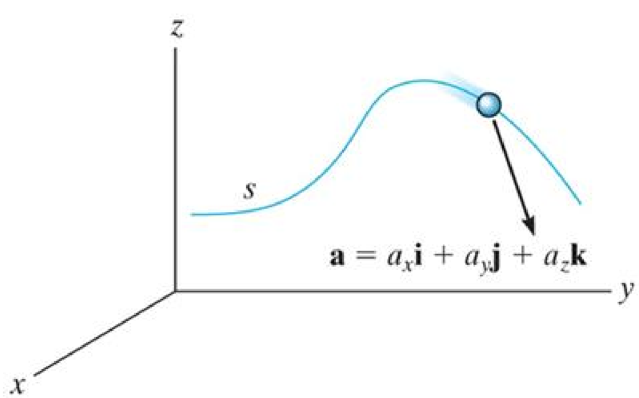
The acceleration vector is the time derivative of the velocity vector (second derivative of the position vector):

$$\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{r}/dt^2 = a_x\,\boldsymbol{i} + a_y\,\boldsymbol{j} + a_z\,\boldsymbol{k}$$

where $a_x = \dot{v}_x = \ddot{x} = dv_x/dt$, $a_y = \dot{v}_y = \ddot{y} = dv_y/dt$,

$a_z = \dot{v}_z = \ddot{z} = dv_z/dt$

The magnitude of the acceleration vector is



Acceleration

$$a = [(a_x)^2 + (a_y)^2 + (a_z)^2\,]^{0.5}$$

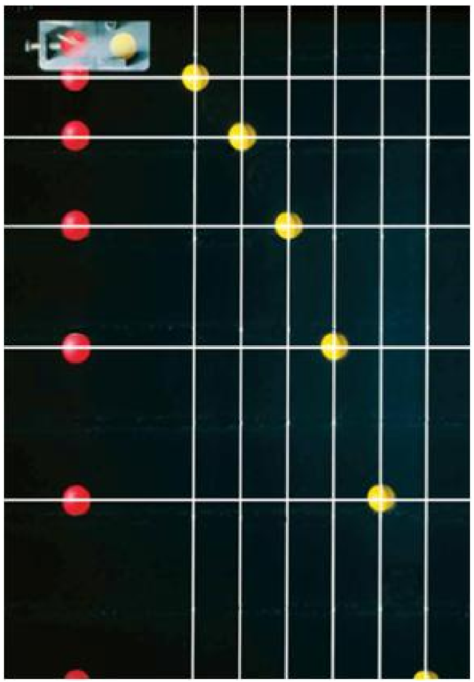
The direction of ***a*** is usually not tangent to the path of the particle.

Motion of a Projectile

Motion of a Projectile

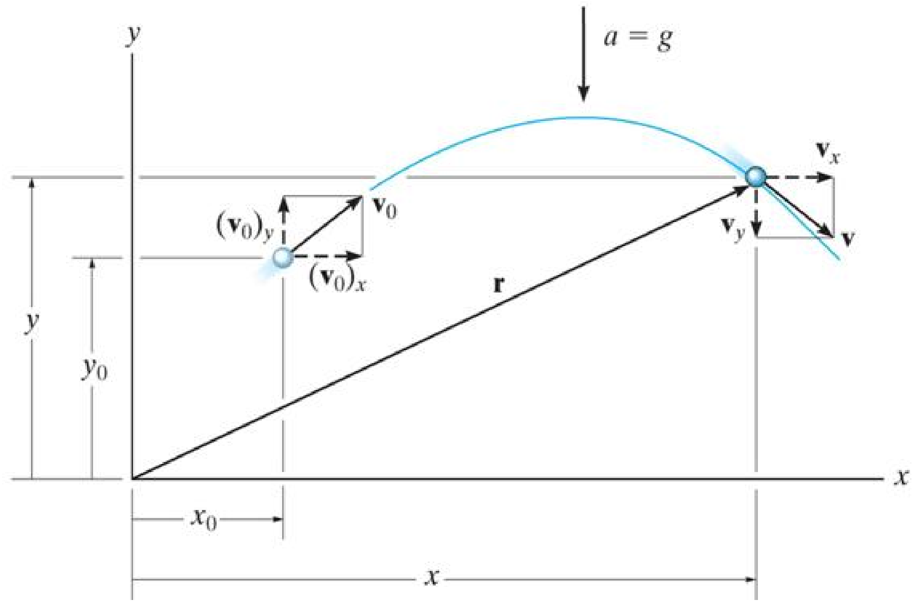
Projectile motion can be treated as two rectilinear motions, one in the horizontal direction experiencing zero acceleration and the other in the vertical direction experiencing constant acceleration (i.e., from gravity).

Motion of a Projectile



For illustration, consider the two balls on the left. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity. Each picture in this sequence is taken after the same time interval. Notice both balls are subjected to the same downward acceleration since they remain at the same elevation at any instant. Also, note that the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction is constant.

Horizontal Motion



Since $a_x = 0$, the velocity in the horizontal direction remains constant ($v_x = v_{ox}$) and the position in the x direction can be determined by:

$$x = x_o + (v_{ox})\ t$$

Why is a_x equal to zero (what assumption must be made if the movement is through the air)?

Vertical Motion

Since the positive y-axis is directed upward, $a_y = -g$.
Application of the constant acceleration equations yields:

$$v_y = v_{oy} - g\,t$$

$$y = y_o + (v_{oy})\,t - \tfrac{1}{2}\,g\,t^2$$

$$v_y^2 = v_{oy}^2 - 2\,g\,(y - y_o)$$

For any given problem, only two of these three equations can be used. Why?

Examples & Questions

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